RESEARCH STATEMENT
O-Yeat Chan

My research interests lie at the confluence of analytic, combinatorial, and experimental number theory. I am especially fascinated by the interplay between analytic and combinatorial techniques in number theory, and the role of modern technology in discovery and proof. Consequently, my research has spanned a diverse range of topics including integer partitions [10], multiple-zeta values [6], special number sequences [11], and numerical evaluation of functions [5]. In fact, because my interests overlap several fields, I have worked with analysts, number theorists, and graph theorists on a variety of projects.

One of my recent papers in combinatorial number theory [11] is an investigation of the Stirling numbers of the second kind $S(n, k)$ modulo prime powers done jointly with D. Manna, an analyst and number theorist at Virginia Wesleyan College. $S(n, k)$ counts the number of ways to partition a set of n elements into exactly k non-empty subsets. The divisibility properties of $S(n, k)$ have applications outside of number theory [15] and their p-adic behaviour show non-trivial structure [1, 8, 13]. In our work, Manna and I applied both numerical and symbolic computer algebra to uncover new theorems on $S(n, k)$ and its ordinary generating function modulo prime powers, which we then proved analytically.

While my research on Stirling numbers is of a combinatorial flavour, my work with D. Borwein and J. Borwein on the evaluation of Bessel functions [5] is more analytic. Motivated by a need to numerically check an assertion in Ramanujan's Lost Notebook involving Bessel functions [4], I searched for new ways of calculating the Bessel functions. The result was a universally convergent evaluation scheme stemming from the integral representations of the Bessel functions.

A more interdisciplinary example of my work is my most recent article [12] with P. Prałat, a graph theorist at West Virginia University, concerning networks with weighted nodes. Consider a graph where each vertex v_i begins with a weight (content) w_i, and at each stage some vertex distributes its contents to its neighbours along its edges, and each edge collects a toll according to the content passing through it. How long does it take for all the edges to recover their (fixed) costs? Our solution in the case of complete graphs is a marriage of analytic and combinatorial approaches: we used analytic methods to estimate the content of each vertex at time t, and combinatorial arguments to show that these estimates are optimal. We were also able to use these methods to obtain order-of-magnitude results for complete bipartite graphs.

I have also been interested in research related to hypergeometric series, polylogarithms, and their evaluations at special values [6, 7]. Results in this area often have implications in physics, as these objects appear in evaluations of high-dimensional integrals with physical interpretations. My article on box integrals [7] with J. Borwein and R. Crandall, a physicist at Reed College, illustrates both the difficulty and the importance of understanding such objects. The box integral $B_n(s)$ is defined by

$$B_n(s) := \int_{[0,1]^n} (x_1^2 + \cdots + x_n^2)^{s/2}\ dx_1 \cdots dx_n.$$

Note that $B_n(1)$ is the expected distance of a random point from any vertex in a unit n-cube, while $B_n(-n+2)$ is the expected electrostatic potential of a point in a unit n-cube with a unit charge at the origin. In [2, 3], D. Bailey, J. Borwein, and R. Crandall showed that $B_n(s)$ have closed form evaluations in terms of polylogarithmic [14] functions when s is an integer and $0 \le n \le 4$. Using a combination of computer-assisted discovery and symbolic computation, Borwein, Crandall, and I were able to extend this result to $n = 5$ for all integral s as well as for certain values of s in higher dimensions. These results not only allow for rapid numerical evaluation of these integrals (a reduction from an n-dimensional object to a finite combination of 1-dimensional objects), but also reveal new insights into the structure of hypergeometric and polylogarithmic sums.

In contrast with the box integrals paper, which required many high-powered tools to relate complex combinations of polylogarithmic sums, my work with J. Borwein on multiple-zeta values [6] applied elementary methods to prove a generalization of a known relation on multidimensional polylogarithmic sums: *the duality relation for multiple-zeta values*. Our proof is inductive, and shows clearly how the inherent duality on the partial sums (the *tails* of the multiple-zeta values) induces duality in higher-dimensional multiple-zetas.

My current research not only includes a continuing investigation into questions related to my past projects, but also new problems that fall within my areas of interest. For example, I am working on another joint with D. Manna on a new q-analogue of the Bernoulli numbers which seem to arise more naturally than the q-analogue described by Carlitz [9]. The analytic properties of these new objects appear to contain a great deal of structure, and a complete description of these objects may reveal new combinatorial insights into the classical Bernoulli numbers as well.

In the long term, I plan to continue studying special functions both as analytic objects which specialize at appropriate values to fundamental mathematical constants, and as combinatorial objects, serving as generating functions for special number sequences. As in the case of the box integrals, many such objects have close connections with theoretical physics and at the same time exhibit relations among important constants in number theory. Often, new results are indicative of deeper structure within a more general setting, perhaps in a higher dimensional analogue or in the introduction of a new parameter such as q-analogues. The functions related to these objects – the zeta functions, the polylogarithmic functions, and more general functions of hypergeometric type – are still very mysterious and there is still much to explore.

References

[1] T. Amdeberhan, D. Manna, and V. H. Moll, *The 2-adic valuation of Stirling numbers*, Experiment. Math. **17** (2008), no. 1, 69–82.

[2] D. H. Bailey, J. M. Borwein, and R. E. Crandall, *Box integrals*, J. Comp. Appl. Math. **206** (2007), 196–208.

[3] D. H. Bailey, J. M. Borwein, and R. E. Crandall, *Advances in the theory of box integrals*, Math. Comput. **79** (2010), 1839–1866.

[4] B. C. Berndt, O-Y. Chan, S.-G. Lim, A. Zaharescu, *Questionable claims found in Ramanujan's Lost Notebook*, in *Tapas in Experimental Mathematics*, T. Amdeberhan and V. Moll, eds., Contemporary Math. **457**, Amer. Math. Soc., (2008) 69–98.

[5] D. Borwein, J. M. Borwein, and O-Y. Chan, *The evaluation of Bessel functions via exp-arc integrals*, J. Math. Anal. Appl. **341** (2008) 478–500.

[6] J. M. Borwein and O-Y. Chan, *Duality in tails of multiple zeta values*, Int. J. Number Theory **6** (2010), no. 3, 501–514.

[7] J. M. Borwein, O-Y. Chan, and R. Crandall, *Higher-dimensional box integrals*, Exp. Math., (2010), 24 pp. To appear.

[8] L. Carlitz, *Congruences for generalized Bell and Stirling numbers*, Duke Math. J. **22** (1955), 193–205.

[9] L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J. **15** (1948), no. 4, 987–1000.

[10] O-Y. Chan, *Some asymptotics for cranks*, Acta Arith. **120** (2005), no. 2, 107–143.

[11] O-Y. Chan and D. V. Manna, *Congruences for Stirling numbers of the second kind*, in *Gems in Experimental Mathematics*, Contemporary Math. **517**, Amer. Math. Soc., (2010), 97–111.

[12] O-Y. Chan and P. Prałat, *Chipping away at the edges: how long does it take?*, (2010), 16 pp. Submitted. Preprint available at `http://www.oyeat.com/pubs.html`

[13] Y. H. Kwong, *Minimum periods of $S(n,k)$ modulo M*, Fibonacci Quart. **27** (1989), 217–221.

[14] L. Lewin, *Polylogarithms and associated functions*, North Holland, 1981.

[15] A. T. Lundell, *A divisibility property for Stirling numbers*, J. Number Theory **10** (1978), 35–54.