MATH 3070

Assignment # 2 Solutions

Due Thursday, September 25, 2008

1. (a) If no term is 2, then suppose we have a prime term. Then it must be odd. So the next term must be even (and therefore composite, as it's not 2) since the common difference is odd and odd + odd = even.

   (b) Suppose we have three consecutive terms p, $p+2$, and $p+4$ such that p is a prime not equal to 3. Then $p \equiv \pm 1 \pmod 3$. If $p \equiv 1 \pmod 3$, then $p+2 \equiv 0 \pmod 3$ and must be composite as $p \geq 2$ and so $p+2 \neq 3$. Similarly, if $p \equiv -1 \pmod 3$ then $p+4 \equiv 0 \pmod 3$ is composite.

   (c) Consider four consecutive terms n, $n+d$, $n+2d$, $n+3d$. If n is odd, then $n+d$ and $n+3d$ are both even and therefore not coprime. If n is even, then n and $n+2d$ are both even and therefore not coprime.

   (d) Let the progression be a, $a+d$, $a+2d$, $\ldots$, $a+999d$, where d is the common difference. Since they are pairwise coprime, for all $1 \leq k < \ell \leq 999$ we have $(a+kd, a+\ell d) = (a+kd, (\ell-k)d) = 1$. If we choose $a = 1$ and $d = 1000!$, then since for any k, $(1+k1000!, 1000!) = 1$, no prime less than 1000 divides $1 + k1000!$. Thus $1 + k(1000!)$ must be coprime to $(\ell - k)1000!$, as $1 \leq \ell - k \leq 999$.

2. Note that $(F_{n+1}, F_n) = (F_n + F_{n-1}, F_n) = (F_n, F_{n-1})$. So since $(F_0, F_1) = 1$, we may induct on n to obtain $(F_{n+1}, F_n) = 1$.

3. Suppose not. Let $2 = p_1 < p_2 < \cdots < p_n$ be the list of all primes congruent to $2 \pmod 3$. Let $N = 3p_2p_3\cdots p_n + 2$. Since $N > p_n$ it must be composite. However, since the product of integers congruent to 0 or $1 \pmod 3$ is either 0 or 1, N must have a prime factor that is congruent to $2 \pmod 3$. But by construction N is coprime to p_i for all $1 \leq i \leq n$, and we have a contradiction.

4. Reduce mod 16. $x^4 \equiv 0, 1 \pmod{16}$ and $y^3 \equiv 0, \pm 1, 8, \pm 5, \pm 3, \pm 7 \pmod{16}$. So $y^6 \equiv 0, 1, 9 \pmod{16}$ and $y^6 + 2 \equiv 2, 3, 11 \pmod{16}$. Since the LHS is never congruent to the RHS, there are no solutions.

5. <u>Solution 1:</u> Note that $2008 = 8(251) = 2^3 + 2^4 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} =$ $2^3(1 + 2(1 + 2(2(1 + 2(1 + 2(1 + 2(1 + 2)\cdots)$. Thus $4^{2008} = (\cdots(4^2 4)^2 4)^2 4)^2 4)^2)^2 4)^2 4)^2)^2)^2$. Successively squaring and reducing yields $4^2 = 16, 64, 64^2 \equiv 96 \equiv -4, -16, (-16)^2 \equiv 56, 24, 24^2 \equiv 76, 4, 4^2 \equiv 16, 16^2 \equiv 56, 24, 24^2 \equiv 76, 4, 16, 56, 56^2 \equiv 36.$. So the last 2 digits are 36.

   <u>Solution 2:</u> Searching for the cycle, we find that the powers of $4 \pmod{100}$ are $4, 16, 64, 56, 24, 96 \equiv -4, -16, -64, -56, -24, 4 \equiv 4^{11}$. So the cycle length is 10 and so $4^{1+10k} \equiv 4 \pmod{100}$. Thus $4^{2001} \equiv 4 \pmod{100}$, so that $4^{2008} \equiv 4^8 \equiv -64 \equiv 36 \pmod{100}$.

6. (a) $3x \equiv 1 \equiv -156 = 3(-52) \pmod{157}$. Thus $x \equiv -52 \equiv 105 \pmod{157}$.

   (b) $7x \equiv 12 \equiv 84 \pmod{36}$. Thus $x \equiv 12 \pmod{36}$.