MATH 3070
Assignment # 1 Solutions
Due Tuesday, September 16, 2008

Prove or disprove the following:

1. For a, $b \in \mathbb{Z}$, if $a|b$ and $b|a$ then $a = b$.

   Solution: False. Let $a = 1$ and $b = -1$. In fact, one can conclude that $a = \pm b$.

2. If x and y are relatively prime, then for any $n \in \mathbb{Z}$ there are integers a and b such that $ax + by = n$.

   Solution: True. By the Euclidean algorithm, there are integers A and B such that $Ax+By = 1$. Multiply through by n to obtain $Anx + Bny = n$. Setting $a = An$ and $b = Bn$ gives the desired linear combination.

3. For any $d \in \mathbb{N}$, if $d|ab$ then either $d|a$ or $d|b$.

   Solution: False. For example, let $a = 2$, $b = 3$, and $d = 6$. Contrast this with Euclid's Lemma.

4. n and $2n + 1$ are always relatively prime.

   Solution: True. $1(2n + 1) + (-2)n = 1$. Apply Corollary 6.3.

5. If $(a, b) = (b, c) = 1$, then $(a, c) = 1$.

   Solution: False. Let $a = c > 1$ and $b = 1$.

6. For all $n \in \mathbb{N}$, there is some positive integer k such that $kn + 1$ is composite.

   True.
   Solution 1: Let n be given. Choose $k = n+2$ so that $kn+1 = n(n+2)+1 = n^2+2n+1 = (n+1)^2$ is composite for $n \geq 1$. (Since it has at least 3 distinct divisors: $1 < n + 1 < (n + 1)^2$).

   Solution 2: Let n be given. We would like to find a positive integer k such that $kn + 1$ is composite, or $kn + 1 = \ell x$, where neither ℓ nor x is ± 1. Rearrange to obtain $kn - \ell x = -1$. Choosing x to be a prime larger than n (possible since there are infinitely many primes) we can find integers a and b such that $an - bx = -1$ using the Euclidean algorithm. As discussed in class, if $a < 0$ we can add multiples of x to a and subtract multiples of n from b to get infinitely many solutions. Thus eventually we will be able to find a solution where $a > 0$ and $b \neq 1$. Using that value of a for k works.

7. For integers a, b, and c, define $\gcd(a, b, c) := \max\{d \in \mathbb{N} : d|a, d|b, \text{and } d|c\}$.
   Then $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$.

   Solution: True. Let $x = \gcd(a, b, c)$, $y = \gcd(a, b)$ and $z = \gcd(y, c)$. We wish to show that $x = z$. Now, by definition $x|a$, $x|b$ and $x|c$. Thus x is a common divisor of a and b and so $x|y$ by Corollary 6.2. But then $x|y$ and $x|c$ implies x is a common divisor of y and c so $x \leq z$. On the other hand, $z = \gcd(y, c)$ implies that $z|c$ and $z|y$. But $z|y$ implies $z|a$ and $z|b$ by transitivity of divisibility. Thus z is a common divisor of a, b, and c, so that $z \leq x$. So we may conclude that $z = x$.

For each of the following pairs (x, y), use the Euclidean Algorithm to find a pair (a, b) such that $ax + by = 1$.

8. $(12, 23)$

   Solution: We divide 23 into 15 to find that

$$\begin{aligned} 23 &= 1 \cdot 12 + 11 \\ 12 &= 1 \cdot 11 + 1. \end{aligned}$$

   Therefore,

$$\begin{aligned} 1 &= 12 - 11 \\ &= 12 - (23 - 12) = 2 \cdot 12 - 23. \end{aligned}$$

   So $a = 2$ and $b = -1$ works.

9. $(15, 44)$

   Solution:

$$\begin{aligned} 44 &= 2 \cdot 15 + 14 \\ 15 &= 14 + 1 \end{aligned}$$

   Therefore,

$$\begin{aligned} 1 &= 15 - 14 \\ &= 15 - (44 - 2 \cdot 15) = 3 \cdot 15 - 44. \end{aligned}$$

   So $a = 3, b = -1$ works.

Finally...

10. Alice and Bob are consultants. Since Alice has a PhD (and Bob doesn't), Bob needs to lower his prices in order to compete. Suppose Alice charges \$273 per hour and Bob charges \$161 per hour. Is it possible for one of them to have made exactly \$14 more than the other some time during the month? If so, how many hours does each have to work? (Assume, of course, that they bill in whole hours only)

    Solution: The question basically asks whether there are integers x, y such that $273x + 161y = 14$. Now, using the Euclidean Algorithm, we find that

$$\begin{aligned} 273 &= 161 + 112 \\ 161 &= 112 + 49 \\ 112 &= 2 \cdot 49 + 14. \end{aligned}$$

    Therefore,

$$\begin{aligned} 14 &= 112 - 2 \cdot 49 \\ &= 112 - 2 \cdot (161 - 112) = 3 \cdot (112) - 2 \cdot 161 \\ &= -2 \cdot 161 + 3 \cdot (273 - 161) = -5 \cdot 161 + 3 \cdot 273. \end{aligned}$$

So if Alice bills 3 hours and Bob bills 5 hours, then Alice will make \$14 more than Bob.

Alternatively, one could carry the gcd calculation to the end, and find a pair x, y such that $273x + 161y = 7$, and then multiply through by 2. That is,

$$\begin{aligned} 49 &= 3 \cdot 14 + 7 \\ 14 &= 2 \cdot 7 + 0. \end{aligned}$$

So that

$$\begin{aligned} 7 &= 49 - 3 \cdot 14 \\ &= 49 - 3 \cdot (112 - 2 \cdot 49) = 7 \cdot 49 - 3 \cdot 112 \\ &= -3 \cdot 112 + 7 \cdot (161 - 112) = -10 \cdot 112 + 7 \cdot 161 \\ &= 7 \cdot 161 - 10 \cdot (273 - 161) = 17 \cdot 161 - 10 \cdot 273. \end{aligned}$$

So if Alice bills 20 hours and Bob bills 34 hours, then Alice will make \$14 *less* than Bob.