MATH 2112 / CSCI 2112
Assignment # 6
Due Wednesday, November 1, 2006

Section 4.1: # 21, 22, 28, 34, 39, 56, 57

Section 4.2: #12, 14

Section 4.3: #21, 22, 25
(Hint: For 25, you need to first prove that $b_n \geq 0$ for all $n \geq 0$.)

Also do these problems:

1. Let $\{a_0, a_1, a_2, \ldots\}$ be a sequence that satisfies the relation

$$a_n = \sqrt{2 + a_{n-1}} \qquad \text{for } n \geq 1.$$

(a) Show that if $a_n < 2$, then $a_{n+1} < 2$.

(b) Show that if $a_n > 2$, then $a_{n+1} > 2$.

(c) Use induction to show that, for all $n \geq 0$,

$$\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{1}}}}}}_{n \text{ 2's}} < 2$$

2. (a) Show that if $n^2 \equiv 2 \pmod 4$, then so is $(n+2)^2$.

(b) Explain why this is not an inductive proof that all positive even integers have squares congruent to $2 \pmod 4$.

3. Prove that, for all $n \geq 1$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

4. On Diabolical Island, the secret hideout of Professor Evil, there is one and only one road and it goes all the way around the island. There are some number of gas stations along the road, but Professor Evil has supplied them in such a way so that the total amount of gas available from all of the stations combined is just enough to go around the island exactly once (in the one and only car on the island). Prove the following:

(a) Regardless of how many gas stations there are in total, there is some gas station along the road that will provide enough gas to reach the next gas station.
(Hint: Prove this by contradiction.)

(b) Regardless of how many gas stations there are in total, there is some gas station along the road where if you begin there with an empty tank of gas, and fill up at every gas station along the way, you will be able to drive all the way around the island.