Dalhousie University
MATH 2002, Summer 2007
Midterm Solutions

1. Determine whether the following vector fields are conservative. If so, find its potential function. If not, state why.

   (a) [10] $\mathbf{F}(x,y) = (e^x + \cos(x+y^2)\ ,\quad 2y\cos(x+y^2) - 9y^2)$.
   Solution:
   We first check whether $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, where $P = e^x + \cos(x+y^2)$ and $Q = 2y\cos(x+y^2) - 9y^2$.
   We calculate:
   $$\frac{\partial P}{\partial y} = -2y\sin(x+y^2), \qquad \frac{\partial Q}{\partial x} = -2y\sin(x+y^2).$$
   So we suspect that $\mathbf{F}(x,y)$ is indeed conservative. Now, we suppose that $\mathbf{F} = \nabla f$ for some f. Then we find that
   $$\frac{\partial f}{\partial x} = P(x,y) = e^x + \cos(x+y^2).$$
   Integrating with respect to x, we obtain $f(x,y) = e^x + \sin(x+y^2) + g(y)$ for some function g that depends only on y. Differentiating with respect to y we obtain
   $$\frac{\partial f}{\partial y} = Q(x,y) = 2y\cos(x+y^2) + g'(y) = 2y\cos(x+y^2) - 9y^2.$$
   Thus we have $g'(y) = -9y^2$ and so $g(y) = -3y^3 + C$. Putting it all together, we find that $\mathbf{F}$ is indeed conservative and its potential function $f(x,y)$ is given by
   $$f(x,y) = e^x + \sin(x+y^2) - 3y^3 + C.$$

   (b) [10] $\mathbf{F}(x,y) = (3x^2y + e^{xy}\ ,\quad x^3 + e^{xy})$.
   Solution: As above, we check whether $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, where $P = 3x^2y + e^{xy}$ and $Q = x^3 + e^{xy}$.
   We calculate:
   $$\frac{\partial P}{\partial y} = 3x^2 + xe^{xy}, \qquad \frac{\partial Q}{\partial x} = 3x^2 + ye^{xy}.$$
   Since they are not equal, we conclude that $\mathbf{F}(x,y)$ is not conservative.

2. [20] (10 points each) For each vector field in the previous problem, evaluate the line integral
   $$\int_C \mathbf{F}\cdot d\mathbf{r}$$
   where C is the part of the hyperbola $xy = 1$ from $(\frac{1}{2}, 2)$ to $(2, \frac{1}{2})$.
   Solution:

   (a) Since $\mathbf{F}(x,y)$ is conservative in this case, we may apply the fundamental theorem for line integrals and conclude that
   $$\int_C \mathbf{F}\cdot d\mathbf{r} = f(2, \tfrac{1}{2}) - f(\tfrac{1}{2}, 2)$$

where $f(x,y) = e^x + \sin(x+y^2) - 3y^3$. Therefore, the value of the line integral is

$$e^2 + \sin(2+1/4) - 3/8 - (e^{1/2} + \sin(1/2+4) - 24) = e^2 - \sqrt{e} + \sin(9/4) - \sin(9/2) + 23\frac{5}{8}.$$

(b) In this case, $\mathbf{F}(x,y)$ is not conservative so we need to parametrize the curve and integrate directly. To that end, let $x = t$ so along $xy = 1$ we have $y = 1/t$. So the curve C is given by

$$C : x = t, \quad y = \frac{1}{t}, \qquad \frac{1}{2} \le t \le 2.$$

From this parametrization we find that $dx = dt$ and $dy = -t^{-2}\ dt$. Thus,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (3x^2y + e^{xy})\ dx + (x^3 + e^{xy})\ dy \\
&= \int_{1/2}^{2} (3t^2/t + e^1)\ dt + (t^3 + e^1)(-t^{-2}\ dt) \\
&= \int_{1/2}^{2} 2t + e - \frac{e}{t^2}\ dt \\
&= t^2 + et + \frac{e}{t}\Big|_{1/2}^{2} \\
&= 4 + 2e + \frac{e}{2} - (1/4 + \frac{e}{2} + 2e) = 4 - \frac{1}{4} = \frac{15}{4}.
\end{aligned}$$

3. [15] Evaluate the integral

$$\iiint_E x^2\ dV$$

where E is the region below the paraboloid $x^2 + y^2 + z = 4$ and above the xy-plane.

Solution: We may consider E as a region that lies under the surface $z = 4 - x^2 - y^2$ and above the domain $x^2 + y^2 \le 4$. So it makes sense to convert to cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Then the region E is given by

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad 0 \le z \le 4 - r^2.$$

Thus we have

$$\begin{aligned}
\iiint_E x^2\ dV &= \int_0^{2\pi} d\theta \int_0^2 r\ dr \int_0^{4-r^2} r^2\cos^2\theta\ dz \\
&= \int_0^{2\pi} \cos^2\theta\ d\theta \int_0^2 r^3\ dr \int_0^{4-r^2} dz \\
&= \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\ d\theta \int_0^2 r^3(4-r^2)\ dr \\
&= \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_{\theta=0}^{2\pi} \left(r^4 - \frac{r^6}{6}\right)\Big|_{r=0}^{2} \\
&= \pi\left(2^4 - \frac{2^6}{6}\right) = \left(16 - \frac{32}{3}\right)\pi = \frac{16\pi}{3}.
\end{aligned}$$

4. [15] Suppose you were hiking in the mountains, and you take a spiral path up to the summit given by the parametric equations

$$x = (20 - t)\cos t, \quad y = (20 - t)\sin t, \quad z = 100t, \qquad 0 \le t \le 20.$$

Calculate the total distance you have travelled on this path.

Solution: We wish to calculate the arc-length of the path given by the parametrization in the problem. That is, we want to evaluate the integral

$$L := \int_0^{20} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

So we calculate:

$$\frac{dx}{dt} = -\cos t - (20 - t)\sin t, \quad \frac{dy}{dt} = -\sin t + (20 - t)\cos t, \quad \frac{dz}{dt} = 100.$$

Thus

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 &= \cos^2 t + 2(20 - t)\sin t\cos t + (20 - t)^2 \sin^2 t \\ \left(\frac{dy}{dt}\right)^2 &= \sin^2 t - 2(20 - t)\sin t\cos t + (20 - t)^2 \cos^2 t \\ \left(\frac{dz}{dt}\right)^2 &= 10000 \end{aligned}$$

Therefore, the arc-length is

$$\begin{aligned} L &= \int_0^{20} \sqrt{1 + (20 - t)^2 + 10000}\, dt \\ &= \int_{20}^{0} -\sqrt{10001 + u^2}\, du \\ &= \frac{u}{2}\sqrt{10001 + u^2} + \frac{10001}{2}\ln(u + \sqrt{10001 + u^2})\Big|_0^{20} \\ &= 200\sqrt{10401} + \frac{10001}{2}\ln(20 + \sqrt{10401}) - \frac{10001}{2}\ln\sqrt{10001}. \end{aligned}$$

5. [15] Evaluate the integral

$$\oint_C 4x\sin y\, dx + 3x^2\cos y\, dy$$

where C is the positively oriented triangle with vertices $(0, 0)$, $(6, 0)$, and $(6, 3)$.

Solution: We apply Green's Theorem to the integral. The domain is the triangle bounded by

C, and is given by $0 \le x \le 6,\ 0 \le y \le x/2$. Thus,

$$\begin{aligned}
\oint_C 4x \sin y \ dx + 3x^2 \cos y \ dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \iint_D (6x \cos y - 4x \cos y) \ dA \\
&= \int_0^6 dx \int_0^{x/2} 2x \cos y \ dy \\
&= \int_0^6 2x \sin(x/2) \ dx \\
&= \int_0^3 8u \sin u \ du \\
&= 8(-u \cos u + \sin u) \Big|_0^3 \\
&= 8 \sin 3 - 24 \cos 3.
\end{aligned}$$

6. Consider the transformation from the xy-plane to the uv-plane given by

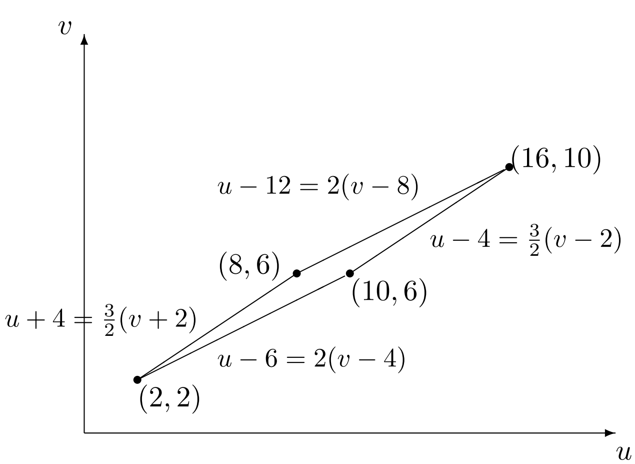
$$u = 3x + 4y, \quad v = 2x + 2y.$$

(a) [8] Let D be the rectangular region $2 \le x \le 4,\ -1 \le y \le 1$. What is the image of D in the uv-plane under this map? Draw a picture and write the equations for the boundary.
Solution: Since the map is linear, the rectangle will map to a parallelogram in the uv-plane. Thus we look at the images of the vertices:

$$(2,-1) \mapsto (2,2), \quad (4,-1) \mapsto (8,6), \quad (4,1) \mapsto (16,10), \quad (2,1) \mapsto (10,6).$$

Along the edges, we have in the xy-plane the lines $x = 2$, $x = 4$, $y = -1$, and $y = 1$. We find their images in the uv-plane:

$$\begin{aligned}
x = 2 &\Rightarrow u = 6 + 4y,\ v = 4 + 2y & &\text{so } u - 6 = 2(v - 4), \\
x = 4 &\Rightarrow u = 12 + 4y,\ v = 8 + 2y & &\text{so } u - 12 = 2(v - 8), \\
y = -1 &\Rightarrow u = 3x - 4,\ v = 2x - 2 & &\text{so } u + 4 = \tfrac{3}{2}(v + 2), \\
y = 1 &\Rightarrow u = 3x + 4,\ v = 2x + 2 & &\text{so } u - 4 = \tfrac{3}{2}(v - 2).
\end{aligned}$$



(b) [4] Calculate the absolute value of the Jacobian $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$.

Solution: We need to solve for x and y in terms of u and v. Multiplying v by 2 and subtracting we find that $x = 2v - u$. Similarly, we find that $2u - 3v = 2y$ so $y = (2u - 3v)/2$. Therefore,

$$\begin{aligned}
\left|\frac{\partial(x,y)}{\partial(u,v)}\right| &= \left|\det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} -1 & 2 \\ 1 & -3/2 \end{pmatrix}\right| = |(3/2 - 2)| \\
&= 1/2.
\end{aligned}$$

(c) [3] What is the Jacobian (in matrix form) of a transformation on 4 variables? (In other words, what is the matrix form of $\dfrac{\partial(w,x,y,z)}{\partial(r,s,t,u)}$?)

Solution:

$$\frac{\partial(w,x,y,z)}{\partial(r,s,t,u)} = \det\begin{pmatrix} \dfrac{\partial w}{\partial r} & \dfrac{\partial w}{\partial s} & \dfrac{\partial w}{\partial t} & \dfrac{\partial w}{\partial u} \\ \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial s} & \dfrac{\partial z}{\partial t} & \dfrac{\partial z}{\partial u} \end{pmatrix}$$

Useful Formulas

Trigonometric Identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
3. $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
4. $\sin 2x = 2 \sin x \cos x$
5. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Integration Formulas

1. $\int u \; dv = uv - \int v \; du$
2. $\int xe^x \; dx = xe^x - e^x + C$
3. $\int \ln x \; dx = x \ln x - x + C$
4. $\int x \sin x \; dx = -x \cos x + \sin x + C$
5. $\int x \cos x \; dx = x \sin x + \cos x + C$
6. $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a} \arctan \dfrac{x}{a} + C$
7. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \arcsin \dfrac{x}{a} + C$
8. $\int \sqrt{a^2 + x^2} \; dx = \dfrac{x}{2}\sqrt{a^2 + x^2} + \dfrac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C$
9. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{a^2 + x^2}) + C$
10. $\int_{-\infty}^{\infty} e^{-x^2} \; dx = \sqrt{\pi}$